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Density of finite gap potentials for the Zakharov–Shabat system

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Abstract. For various spaces of potentials we prove that the set of finite gap potentials of the Zakharov–Shabat system is dense. In particular our result holds for Sobolev spaces and for spaces of analytic potentials of a given type.

1. Introduction

This paper is an addendum to our investigation of spectral properties of the Zakharov–Shabat operator

$$L(\psi) := \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & \psi_1 \\ \psi_2 & 0 \end{pmatrix}$$

initiated in [6,5] (cf. also [4]). Its aim is to establish the density of finite gap potentials in weighted Sobolev spaces. Here $\psi = (\psi_1, \psi_2)$ and the components ψ_1 and ψ_2 are 1-periodic functions in the weighted Sobolev space $H^w \equiv H^w_{\mathbb{C}}$ of 2-periodic functions

$$H^w := \left\{ f \mid f(x) := \sum_{-\infty}^{\infty} \hat{f}(k)\, \mathrm{e}^{\mathrm{i}\pi k x};\ \|f\|_w < \infty \right\}$$

with

$$\|f\|_w := \left(\sum_{k\in\mathbb{Z}} w(k)^2 \bigl|\hat{f}(k)\bigr|^2 \right)^{1/2}$$

and $w = (w(k))_{k\in\mathbb{Z}}$ is a weight, i.e., a sequence of positive numbers with $w(k) \geqslant 1$, $w(-k) = w(k)$ and $w(k) \leqslant w(k-j)w(j)\ \forall j,k \in \mathbb{Z}$. Denote by $(\lambda_n^\pm(\psi))_{n\in\mathbb{Z}}$ the periodic eigenvalues of $L(\psi)$ when considered on the interval $[0,2]$ (cf. [5] for a detailed description of the periodic spectrum). As usual, $(\lambda_n^\pm(\psi))_{n\in\mathbb{Z}}$ is arranged in lexicographic ordering. The potential ψ is said to be a *finite gap potential* if $\{n \in \mathbb{Z} \mid \lambda_n^+(\psi) \neq \lambda_n^-(\psi)\}$ is *finite*. Recall from [5] that a weight w is said to be a δ-*weight* ($\delta > 0$) if $w_*(k) := (1+|k|)^{-\delta} w(k)$ is a weight as well. The potential $\psi = (\psi_1, \psi_2)$ is said to be of real type if $\psi_1 = \overline{\psi_2}$. In such a case the operator $L(\psi)$ is selfadjoint.

Theorem 1.1. *Let w be a δ-weight for some $\delta > 0$. For any $\varepsilon > 0$ and any 1-periodic potential $\psi = (\psi_1, \psi_2)$ in $H^w \times H^w$ there is a finite gap potential $\psi_\varepsilon = (\psi_{\varepsilon 1}, \psi_{\varepsilon 2})$ in $H^w \times H^w$ at distance at most ε from ψ, i.e., $\sup_{1 \leqslant j \leqslant 2} \|\psi_{\varepsilon j} - \psi_j\|_w < \varepsilon$, so that*

$$\lambda_n^+(\psi_\varepsilon) = \lambda_n^-(\psi_\varepsilon), \quad \forall |n| \geqslant N_\varepsilon,$$

where $N_\varepsilon \equiv N_\varepsilon(\psi) \geqslant 1$.

If ψ is of real type, i.e., $\psi_1 = \overline{\psi_2}$, then ψ_ε can be chosen to be of real type as well.

Remark 1.2. The number N_ε in Theorem 1.1 can be chosen uniformly on compact sets of 1-periodic potentials in H^w.

Remark 1.3. Since $H^w \times H^w$ is dense in $L^2([0,1]; \mathbb{C}^2)$ for any δ-weight w Theorem 1.1 implies that finite gap potentials are dense in $L^2([0,1]; \mathbb{C}^2)$.

In the special case where ψ is of real type and an element of the Sobolev space H^{w_p} with w_p denoting the Sobolev weight $w_p(k) := (1 + |k|)^p$ $(k \in \mathbb{Z})$ and $p \in \mathbb{Z}_{\geqslant 0}$, Theorem 1.1 has been proved in [10] and, independently in [3], using results established in [9] and [2] respectively.

Similar results as the ones presented here for the Zakharov–Shabat operator $L(\psi_1, \psi_2)$ have been obtained previously for the Hill operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ in [11].

To prove Theorem 1.1 (cf. [5] for a discussion of the spaces H^w) we follow the approach used in [11]: as a set-up we take the Fourier bloc decomposition introduced first for the Hill operator in [7,8] and worked out subsequently for the Zakharov–Shabat operator in [6,5]. Unlike in [11] where a contraction mapping argument was used to obtain the density results for the Hill operator, we get a short proof of Theorem 1.1 by applying the inverse function theorem in a straightforward way. As in [11] the main feature of the presented proof is that it does not involve any results from inverse spectral theory (cf. [1] for related results for the Hill equation). Throughout the remainder of this paper we use the notation introduced in [5].

2. Proof of Theorem 1.1

Note that for any two weights w, w' with $w \leqslant w'$, $H^{w'}$ is a dense subspace of H^w. Hence without loss of any generality we may assume that w is a δ-weight with

$$\delta = \frac{1}{2}.$$

As explained in [5, Section 2.6], for $M \geqslant 1$ arbitrary, there exists $N \equiv N_M \geqslant 1$ so that for any $\psi = (\psi_1, \psi_2)$ with

$$\|\psi\|_w := \sup_{j=1,2} \|\psi_j\|_w \leqslant M$$

the eigenvalues $\lambda_n^\pm(\psi)$ for $|n| \geqslant N_M$ are of the form

$$\lambda_n^\pm(\psi) = n\pi + z_n^\pm, \quad |z_n^\pm| < \frac{\pi}{4},$$

where the complex numbers $z_n^\pm \equiv z_n^\pm(\psi)$ are the two solutions in $|z| < \pi/4$ of the following system

$$z = \alpha(n,z) + \zeta, \tag{2.1}$$

$$\zeta^2 - \big(\hat{\psi}_2(2n) + \beta^+(n,z)\big)\big(\hat{\psi}_1(-2n) + \beta^-(n,z)\big) = 0 \tag{2.2}$$

with $\alpha(n,z) \equiv \alpha(n,z,\psi)$ and $\beta^\pm(n,z) \equiv \beta^\pm(n,z,\psi)$ given by (cf. [5, (2.23)–(2.26), (2.29)])

$$\alpha(n,z) := \langle S^n\hat{\psi}_2, (z-D_n)^{-1}(\mathrm{Id}-Q_n)^{-1}S^nJ\hat{\psi}_1\rangle, \tag{2.3}$$

$$\beta^+(n,z) := \langle S^n\hat{\psi}_2, (z-D_n)^{-1}R_n^{(1)}(\mathrm{Id}-P_n)^{-1}S^n\hat{\psi}_2\rangle, \tag{2.4}$$

$$\beta^-(n,z) := \langle S^nJ\hat{\psi}_1, (z-D_n)^{-1}R_n^{(2)}(\mathrm{Id}-Q_n)^{-1}S^nJ\hat{\psi}_1\rangle. \tag{2.5}$$

Here S and J denote the shift respectively involution operator given by

$$S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad Sa(k) := a(k+1), \quad \forall k \in \mathbb{Z},$$
$$J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad Ja(k) := a(-k), \quad \forall k \in \mathbb{Z},$$

and, for any $n \in \mathbb{Z}$, $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{\mathbb{Z}\backslash n}$ denotes the bilinear form (*no* complex conjugation)

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle := \sum_{k\neq n} \big(a(k)c(k) + b(k)d(k)\big).$$

Finally, $D_n,\ P_n,\ Q_n,\ R_n^{(1)}$ and $R_n^{(2)}$ are operators on $\ell^2(\mathbb{Z}\backslash n)$ defined by

$$D_n := \big((k-n)\pi\delta_{kj}\big)_{k,j\in\mathbb{Z}\backslash n}, \qquad P_n := R_n^{(2)}R_n^{(1)}, \qquad Q_n := R_n^{(1)}R_n^{(2)}$$

and

$$R_n^{(1)}a := J\big(\hat{\psi}_1 * (z-D_n)^{-1}\big)a, \qquad R_n^{(2)}a := \hat{\psi}_2 * J(z-D_n)^{-1}a.$$

By (2.1), $z_n^+ = z_n^-$ whenever $\zeta_n^+ = 0$ and $\zeta_n^- = 0$. Hence by (2.2) (cf. [5, Proposition 2.12]), a sufficient condition for $\lambda_n^+(\psi) = \lambda_n^-(\psi)$ is that there exists z_n with $|z_n| < \pi/4$ so that

$$\hat{\psi}_2(2n) + \beta^+(n,z_n,\psi) = 0, \tag{2.6}$$

$$z_n - \alpha(n,z_n,\psi) = 0. \tag{2.7}$$

This suggests a way how to prove Theorem 1.1: given $\varepsilon > 0$ and a 1-periodic potential $\psi \in H^w \times H^w$ with $\|\psi\|_w \leqslant M/4$ we want to find $N_\varepsilon \equiv N_\varepsilon(\psi) \geqslant N_M$ so that there exists a 1-periodic potential ψ_ε in $H^w \times H^w$ with $\|\psi_\varepsilon - \psi\|_w \leqslant \varepsilon$ and $(z_n)_{|n|\geqslant N_\varepsilon}$ with $|z_n| < \pi/4$ satisfying the system of Eqs (2.6), (2.7). For the appropriate set-up of this system we review some properties of the coefficients $\alpha(n,z)$ and $\beta^+(n,z)$ established in [5].

First let us introduce some more notation. Denote by $\mathcal{D}_r \subseteq \mathbb{C}$ the open disc, $\mathcal{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$ and by B_M^w the open ball,

$$B_M^w := \{\psi = (\psi_1, \psi_2) \mid \psi_j \in H^w \text{ 1-periodic; } \|\psi_j\|_w < M,\ j = 1, 2\}.$$

Concerning the coefficient $\alpha(n, z, \psi)$, recall from [5] that for any $|n| \geqslant N_M$, $\alpha(n, z, \psi)$ is defined for $(z, \psi) \in \mathcal{D}_{\pi/4} \times B_M^w$. From the definition of $\alpha(n, z, \psi)$ it is straightforward to see that it is analytic on $\mathcal{D}_{\pi/4} \times B_M^w$.

Recall the following Lemma 2.7 of [5]. (Note that we assume throughout that w is a δ-weight with $\delta = 1/2$.)

Lemma 2.1. *For any $(z, \psi) \in \mathcal{D}_{\pi/4} \times B_M^w$ and any $|n| \geqslant N_M$,*

$$\bigl|\alpha(n, z, \psi)\bigr| \leqslant \frac{4M^2}{\langle n \rangle}.$$

By Cauchy's estimate, Lemma 2.1 leads to the following estimates of the differentials $\mathrm{d}_\psi \alpha$ and $\mathrm{d}_z \alpha$.

Corollary 2.2. *For any $(z, \psi) \in \mathcal{D}_{\pi/8} \times B_{M/2}^w$ and any $|n| \geqslant N_M$*

$$\bigl\|\mathrm{d}_\psi \alpha(n, z, \psi)\bigr\| \leqslant \frac{16M}{\langle n \rangle}, \qquad \bigl|\mathrm{d}_z \alpha(n, z, \psi)\bigr| \leqslant \frac{32M^2}{\pi} \frac{1}{\langle n \rangle}.$$

To state the next result denote by

$$\mathcal{D}_{\pi/8}^\infty \equiv \mathcal{D}_{\pi/8}^\infty\bigl(\mathbb{Z} \setminus [-N_M, N_M]\bigr) \subseteq \ell^\infty\bigl(\mathbb{Z} \setminus [-N_M, N_M]; \mathbb{C}\bigr)$$

the open ball of radius $\pi/8$ centered at 0 in the Banach space $\ell^\infty(\mathbb{Z} \setminus [-N_M, N_M]; \mathbb{C})$.

Proposition 2.3. *The map*

$$\begin{aligned} &\mathcal{D}_{\pi/8}^\infty \times B_{M/2}^w \to \ell^2\bigl(\mathbb{Z} \setminus [-N_M, N_M], \mathbb{C}\bigr), \\ &\bigl((z_n)_{|n|>N_M}, \psi\bigr) \mapsto \bigl(\alpha(n, z_n, \psi)\bigr)_{|n|>N_M} \end{aligned}$$

is analytic and satisfies for any $N \geqslant N_M$

$$\begin{aligned} &\bigl\|\bigl(\alpha(n, z_n, \psi)\bigr)_{|n|>N}\bigr\| \leqslant 2M^2 N^{-1/2}, \\ &\bigl\|d_\psi \bigl(\alpha(n, z_n, \psi)\bigr)_{|n|>N}\bigr\| \leqslant 32MN^{-1/2} \end{aligned}$$

and, with $Z := (z_n)_{|n|>N}$,

$$\bigl\|d_Z \bigl(\alpha(n, z_n, \psi)\bigr)_{|n|>N}\bigr\| = \bigl\|\bigl(\delta_{nk}\bigl(\partial_{z_k} \alpha(n, z_n, V)\bigr)_{|n|,|k|>N}\bigr)\bigr\| \leqslant \frac{32M^2}{\pi} N^{-1}.$$

Concerning the coefficient $\beta^+(n,z,\psi)$ recall from [5] that for any $|n| \geqslant N_M$, $\beta^+(n,z,\psi)$ is well defined for $(z,V) \in \mathcal{D}_{\pi/4} \times B_M^w$.

From its definition, it is straightforward to see that $\beta^+(n,z,\psi)$ is analytic on $\mathcal{D}_{\pi/4} \times B_M^w$. Recall the following Proposition 2.10 in [5]. (Again note that we have chosen $\delta = 1/2$, hence $\delta_* = 1/2$.)

Lemma 2.4. *There exists $C \geqslant 1$ so that for any $\psi \in B_M^w$,*

$$\sum_{|n|>N_M} \langle n\rangle^2 w(2n)^2 \sup_{|z|\leqslant \pi/4} \left|\beta^+(n,z,\psi)\right|^2 \leqslant C.$$

Again by Cauchy's estimate we obtain estimates for the derivatives of $\beta^+(n,z,\psi)$. To state them recall that ℓ_w^2 denotes the ℓ^2-Hilbert space with weight w and w_1 denotes the weight

$$w(k) := \langle k\rangle w(k) \quad (\forall k \in \mathbb{Z}).$$

Proposition 2.5. *The map*

$$\mathcal{D}_{\pi/8}^\infty \times B_{M/2}^w \to \ell_{w_1}^2(\mathbb{Z} \setminus [-N_M, N_M]; \mathbb{C}),$$
$$\big((z_n)_{|n|>N_M}, \psi\big) \mapsto \big(\beta^+(n,z_n,\psi)\big)_{|n|>N_M}$$

is analytic and satisfies for any $N \geqslant N_M$, with $C \geqslant 1$ given as in Lemma 2.4*,*

(i) $\|(\beta^+(n,z_n,\psi))_{|n|>N}\|_w \leqslant 2CN^{-1/2}$,
(ii) $\|d_\psi(\beta^+(n,z_n,\psi))_{|n|>N}\|_w \leqslant (4C/M)N^{-1/2}$
and for any $Z := (z_n)_{|n|>N} \in \mathcal{D}_{\pi/8}^\infty$
(iii) $\|d_Z(\beta^+(n,z_n,\psi))_{|n|>N}\|_w \leqslant (16C/\pi)N^{-1/2}$.

It turns out that a convenient set-up for the system of Eqs (2.6), (2.7) is the following one: introduce for $N \geqslant N_M$

$$E \equiv E_N := \left\{v = (v_n)_{|n|>N} \mid \|v\|_w < \frac{M}{4}\right\},$$

where $\|v\|_w = (\sum_{|n|>N} w(2n)^2 |v_n|^2)^{1/2}$ and

$$\mathcal{Z} \equiv \mathcal{Z}_N := \left\{Z = (z_n)_{|n|>N} \mid \|Z\| = \bigg(\sum_{|n|>N} |z_n|^2\bigg)^{1/2} < \frac{\pi}{8}\right\}.$$

For any $\psi = (\psi_1, \psi_2) \in B_{M/4}^w$ and any $v \in E_N$ define $\psi^v = (\psi_1^v, \psi_2^v)$ by

$$\psi_2^v(x) := \sum_{|n|\leqslant N} \hat{\psi}_2(2n)\,\mathrm{e}^{\mathrm{i}2\pi nx} + \sum_{|n|>N} v_n\,\mathrm{e}^{\mathrm{i}2\pi nx} \tag{2.8}$$

and

$$\psi_1^v(x) := \sum_{|n|\leqslant N} \hat{\psi}_1(2n)\, \mathrm{e}^{\mathrm{i}2\pi nx} + \sum_{|n|>N} \overline{v}_{-n}\, \mathrm{e}^{\mathrm{i}2\pi nx}. \tag{2.9}$$

The definition of ψ_1^v has been chosen in such a way that ψ^v is of real type if ψ is of real type. Notice that for $v \in E$ and $\psi \in B^w_{M/4}$, $\|\psi_j^v\|_w \leqslant M/2$ for $j = 1, 2$ and hence $\psi^v \in B^w_{M/2}$. Finally, introduce for any given $\psi \in B^w_M$ and $N \geqslant N_M$ a map $\Lambda \equiv \Lambda_\psi^{(N)}$ defined for any $(v, Z) \in E \times \mathcal{Z}$ by $\Lambda(v, Z) := (\Lambda_n(v, Z))_{|n|>N}$ with

$$\Lambda_n(v, Z) := \big(v_n + \beta^+\big(n, z_n, \psi^v\big), z_n - \alpha\big(n, z_n, \psi^v\big)\big).$$

By Propositions 2.3 and 2.5, the range of $\Lambda^{(N)}$ is contained in $\ell^2_w(\mathbb{Z} \setminus [-N, N]; \mathbb{C}) \times \ell^2(\mathbb{Z} \setminus [-N, N]; \mathbb{C})$ and is analytic. Choose on $\ell^2_w \times \ell^2$ the norm defined for any $(v, Z) \in \ell^2_w \times \ell^2$ by

$$\big\|(v, Z)\big\| = \max\big(\|v\|_w, \|Z\|\big).$$

Lemma 2.6. *For any $M \geqslant 1$ there exists $N \geqslant N_M$ so that for any $\psi \in B^w_{M/4}$ and any $(v, Z) \in E_N \times \mathcal{Z}_N$,*

$$\big\|d_{v,Z}\Lambda_\psi^{(N)} - \mathrm{Id}^{(N)}\big\| \leqslant \frac{1}{2},$$

where $\mathrm{Id}^{(N)}$ denotes the identity map on $E_N \times \mathcal{Z}_N$.

Proof. The differential $d_{v,Z}\Lambda^{(N)}$ takes the form

$$d_{v,Z}\Lambda^{(N)} = \begin{pmatrix} \mathrm{Id}_E + d_v(\beta_n^+)_{|n|>N} & d_Z(\beta_n^+)_{|n|>N} \\ -d_v(\alpha_n)_{|n|>N} & \mathrm{Id}_{\mathcal{Z}} - d_Z(\alpha_n)_{|n|>N} \end{pmatrix}.$$

Hence by Propositions 2.3 and 2.5 there exists $N \geqslant N_M$ so that for any $\psi \in B^w_{M/4}$ and $(v, Z) \in E \times \mathcal{Z}$,

$$\begin{aligned} &\|d_{v,Z}\Lambda^{(N)} - \mathrm{Id}^{(N)}\|_w \\ &\quad\leqslant \big\|d_v\big(\beta_n^+\big)_{|n|>N}\big\|_w + \big\|d_Z\big(\beta_n^+\big)_{|n|>N}\big\|_w + \big\|d_v(\alpha_n)_{|n|>N}\big\| + \big\|d_Z(\alpha_n)_{|n|>N}\big\| \leqslant 1/2. \quad \square \end{aligned}$$

Proof of Theorem 1.1. Let $0 < \varepsilon \leqslant 1$ and a 1-periodic potential ψ in H^w. Let $M > 4\|\psi\|_w$ and choose $N_1 \geqslant N$ with N as in Lemma 2.6 so that

$$\Big(\sum_{|n|>N_1} w(2n)^2 |\hat{\psi}_1(2n)|^2\Big)^{1/2} < \frac{\varepsilon}{2} \quad \text{and} \quad \big\|Z^{(0)}(\psi)\big\| < \pi/8, \tag{2.10}$$

where

$$Z^{(0)} \equiv Z^{(0)}(\psi) := \big(\lambda_n^+(\psi) - n\pi\big)_{|n|>N_1}.$$

Further define

$$v^{(0)} \equiv v^{(0)}(\psi) := \left(\hat{\psi}_2(2n)\right)_{|n|>N_1}.$$

As $\|v^{(0)}\|_w \leqslant \|\psi\|_w \leqslant M/4$ it follows that $(v^{(0)}, Z^{(0)}) \in E_{N_1} \times \mathcal{Z}_{N_1}$. Hence, by Lemma 2.6, $d_{v^{(0)},Z^{(0)}} \Lambda_\psi^{(N_1)}$ is invertible. By the inverse function theorem there is $\eta > 0$ so that one can find an open neighborhood U_1 of $(v^{(0)}, Z^{(0)}) \equiv (v^{(0)}(\psi), Z^{(0)}(\psi))$ contained in the open ball $B_{\varepsilon/2}(v^{(0)}, Z^{(0)})$ intersected with $E_{N_1} \times \mathcal{Z}_{N_1}$ so that $\Lambda_\psi^{(N_1)}$ is a diffeomorphism from U_1 onto the ball $B_\eta(\Lambda_\psi^{(N_1)}(v^{(0)}, Z^{(0)}))$ in

$$\ell^2_w(\mathbb{Z} \setminus [-N_1, N_1];\ \mathbb{C}) \times \ell^2(\mathbb{Z} \times [-N_1, N_1]; \mathbb{C}).$$

Choose $N_2 \equiv N_2(\psi)$ with $N_2 \geqslant N_1$ so that

$$\|\Lambda_\psi^{(N_2)}(v^{(0)}, Z^{(0)})\| < \eta.$$

Then

$$\varrho := \left(\left(\Lambda_n(v^{(0)}, Z^{(0)}\right)_{N_1<|n|\leqslant N_2},\ (0,0)_{|n|>N_2}\right)$$

is an element in $B_\eta(\Lambda_\psi^{(N_1)}(v^{(0)}, Z^{(0)}))$. Hence there exists $(v, Z) \in U_1$ so that

$$\varrho = \Lambda_\psi^{(N_1)}(v, Z).$$

In particular, for any $|n| > N_2$

$$v_n + \beta^+(n, z_n, \psi^v) = 0, \qquad z_n - \alpha(n, z_n, \psi^v) = 0,$$

where ψ^v is defined as in (2.8), (2.9) with N given by N_1. This means that Eqs (2.6), (2.7) are satisfied for ψ^v and $(z_n)_{|n|>N_2}$, hence ψ^v is a finite gap potential, or more precisely,

$$\lambda_n^+(\psi^v) = \lambda_n^-(\psi^v), \quad \forall |n| > N_2.$$

Furthermore, as $(v, Z) \in U_1 \subset B_{\varepsilon/2}(v^{(0)}, Z^{(0)})$ we have

$$\|\psi_2 - \psi_2^v\|_w = \left(\sum_{|n|>N_1} w(2n)^2 |v_n - v_n^0|^2\right)^{1/2} < \varepsilon/2$$

and, by the definition (2.9) of ψ_1^v and the estimate (2.10),

$$\|\psi_1 - \psi_1^v\|_w \leqslant \left(\sum_{|n|>N_1} w(2n)^2 \left|\hat{\psi}_1(2n)\right|^2\right)^{1/2} + \left(\sum_{|n|>N_1} w(2n)^2 |v_n|^2\right)^{1/2} < \varepsilon$$

hence

$$\|\psi - \psi^v\|_w = \sup_{1\leqslant j\leqslant 2} \|\psi_j - \psi_j^v\| < \varepsilon.$$

This shows the claimed result with $\psi_\varepsilon := \psi^v$ and $N_\varepsilon := N_2$ for an arbitrary potential $\psi \in H^w$. If ψ is of real type $\psi_\varepsilon = \psi^v$ is of real type as well by the definition (2.8), (2.9) of ψ_v. $\square$

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